

Linear-Vertex Kernel for the Problem of Packing r -Stars into a Graph without Long Induced Paths

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Abstract

Let integers $r \geq 2$ and $d \geq 3$ be fixed. Let \mathcal{G}_d be the set of graphs with no induced path on d vertices. We study the problem of packing k vertex-disjoint copies of $K_{1,r}$ ($k \geq 2$) into a graph G from parameterized preprocessing, i.e., kernelization, point of view. We show that every graph $G \in \mathcal{G}_d$ can be reduced, in polynomial time, to a graph $G' \in \mathcal{G}_d$ with $O(k)$ vertices such that G has at least k vertex-disjoint copies of $K_{1,r}$ if and only if G' has. Such a result is known for arbitrary graphs G when $r = 2$ and we conjecture that it holds for every $r \geq 2$.

1 Introduction

For a fixed graph H , the problem of deciding whether a graph G has k vertex-disjoint copies of H is called H -PACKING. The problem has many applications (see, e.g., [1, 2, 10]), but unfortunately it is almost always intractable. Indeed, Kirkpatrick and Hell [10] proved that if H contains a component with at least three vertices then H -PACKING is NP-complete. Thus, approximation, parameterised, and exponential algorithms have been studied for H -PACKING when H is a fixed graph, see, e.g., [1, 6, 7, 13, 14].

In this note, we will consider H -PACKING when $H = K_{1,r}$ and study $K_{1,r}$ -PACKING from parameterized preprocessing, i.e., kernelization, point of view.¹ Here k is the parameter. As a parameterized problem, $K_{1,r}$ -PACKING was first considered by Prieto and Sloper [13] who obtained an $O(k^2)$ -vertex kernel for each $r \geq 2$ and a kernel with at most $15k$ vertices for $r = 2$. (Since the case $r = 1$ is polynomial-time solvable, we may restrict ourselves to $r \geq 2$.) The same result for $r = 2$ was proved by Fellows *et al.* [6] and it was improved to $7k$ by Wang *et al.* [14].

Fellows *et al.* [6] note that, using their approach, the bound of [13] on the number of vertices in a kernel for any $r \geq 3$ can likely be improved to subquadratic. We believe that, in fact, there is a linear-vertex kernel for every $r \geq 3$ and we prove Theorem 1 to support our conjecture. A path P in a graph G , is called *induced* if it is an induced subgraph of G . For an integer $d \geq 3$, let \mathcal{G}_d denote the set of all graphs with no induced path on d vertices.

Theorem 1. *Let integers $r \geq 2$ and $d \geq 3$ be fixed. Then $K_{1,r}$ -PACKING restricted to graphs in \mathcal{G}_d , has a kernel with $O(k)$ vertices.*

Since d can be an arbitrary integer larger than two, Theorem 1 is on an ever increasing class of graphs which, in the “limit”, coincides with all graphs. To show that Theorem 1 is an optimal² result, in a sense, we prove that $K_{1,r}$ -PACKING restricted to graphs in \mathcal{G}_d is \mathcal{NP} -hard already for $d = 5$ and every fixed $r \geq 3$:

Theorem 2. *Let $r \geq 3$. It is \mathcal{NP} -hard to decide if the vertex set of a graph in \mathcal{G}_5 can be partitioned into vertex-disjoint copies of $K_{1,r}$.*

We cannot replace \mathcal{G}_5 by \mathcal{G}_4 (unless $\mathcal{NP} = \mathcal{P}$) due to the following assertion, whose proof is given in the Appendix.

Theorem 3. *Let $r \geq 3$ and $G \in \mathcal{G}_4$. We can find the maximal number of vertex-disjoint copies of $K_{1,r}$ in G in polynomial time.*

2 Terminology and Notation

For a graph G , $V(G)$ ($E(G)$, respectively) denotes the vertex set (edge set, respectively) of G , $\Delta(G)$ denotes the maximum degree of G and n its number of vertices. For a vertex u and a vertex set X in G , $N(u) = \{v : uv \in E(G)\}$, $N[u] = N(u) \cup \{u\}$, $d(u) = |N(u)|$, $N_X(u) = N(u) \cap X$, $d_X(u) = |N_X(u)|$

¹We provide basic definitions on parameterized algorithms and kernelization in the next section, for recent monographs, see [4, 5]; [11, 12] are recent survey papers on kernelization.

²If $K_{1,r}$ -PACKING was polynomial time solvable, then it would have a kernel with $O(1)$ vertices.

and $G[X]$ is the subgraph of G induced by X . We call $K_{1,r}$ an r -star. We say a star *intersects* a vertex set if the star uses a vertex in the set. We use (G, k, r) to denote an instance of the r -star packing problem. If there are k vertex-disjoint r -stars in G , we say (G, k, r) is a YES-instance, and we write $G \in \star(k, r)$. Given disjoint vertex sets S, T and integers s, r , we say that S has s r -stars in T if there are s vertex-disjoint r -stars with centers in S and leaves in T .

A *parameterized problem* is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ . A parameterized problem L is *fixed-parameter tractable* if the membership of an instance (I, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k)|I|^{O(1)}$ where f is a computable function of the *parameter* k only. Given a parameterized problem L , a *kernelization* of L is a polynomial-time algorithm that maps an instance (x, k) to an instance (x', k') (the *kernel*) such that $(x, k) \in L$ if and only if $(x', k') \in L$ and $k' + |x'| \leq g(k)$ for some function g . It is well-known that a decidable parameterized problem L is fixed-parameter tractable if and only if it has a kernel. Kernels of small size are of main interest, due to applications.

3 Proof of Theorem 1

Note that the 1-star packing problem is the classic maximum matching problem and if $k = 1$, the r -star packing problem is equivalent to deciding whether $\Delta(G) \geq r$. Both of these problems can be solved in polynomial time. Henceforth, we assume $r, k > 1$.

A vertex u is called a *small vertex* if $\max\{d(v) : v \in N[u]\} < r$. A graph without a small vertex is a *simplified graph*.

We now give two reduction rules for an instance (G, k, r) of $K_{1,r}$ -PACKING.

Reduction Rule 1. *If graph G contains a small vertex v , then return the instance $(G - v, k, r)$.*

It is easy to observe that Reduction Rule 1 can be applied in polynomial time.

Reduction Rule 2. *Let $G = (V, E)$ be a graph and let C, L be two vertex-disjoint subsets of V . The pair (C, L) is called a *constellation* if $G[C \cup L] \in \star(|C|, r)$ and there is no star $K_{1,r}$ intersecting L in the graph $G[V \setminus C]$. If (C, L) is a constellation, return the instance $(G[V \setminus (C \cup L)], k - |C|)$.*

It is easy to observe that Reduction Rule 2 can be applied in polynomial time, provided we are given a suitable constellation.

Lemma 1. *Reduction Rules 1 and 2 are safe.*

Proof. Clearly, a small vertex v can not appear in any r -star. Therefore Reduction Rule 1 is safe as G and $G - v$ will contain the same number of r -stars.

To see that Reduction Rule 2 is safe, it is sufficient to show that $G \in \star(k, r)$ if and only if $G[V \setminus (C \cup L)] \in \star(k - |C|, r)$. On the one hand, if $G[V \setminus (C \cup L)] \in \star(k - |C|, r)$, the hypothesis $G[C \cup L] \in \star(|C|, r)$ implies $G \in \star(k, r)$. On the other hand, there are at most $|C|$ vertex-disjoint stars intersecting C . But by hypothesis, every star intersecting L also intersects C . We deduce that there are at most $|C|$ stars intersecting $C \cup L$, and so if $G \in \star(k, r)$, there are at least $k - |C|$ stars in $G[V - (C \cup L)]$: $G[V \setminus (C \cup L)] \in \star(k - |C|, r)$. \square

Note that as both rules modify a graph by deleting vertices, any graph G' that is derived from a graph $G \in \mathcal{G}_d$ by an application of Rules 1 or 2 is also in \mathcal{G}_d .

Recall the Expansion Lemma, which is a generalization of the well-known Hall's theorem.

Lemma 2. (*Expansion Lemma*)[8] *Let r be a positive integer, and let m be the size of the maximum matching in a bipartite graph G with vertex bipartition $X \cup Y$. If $|Y| > rm$, and there are no isolated vertices in Y , then there exist nonempty vertex sets $S \subseteq X, T \subseteq Y$ such that S has $|S|$ r -stars in T and no vertex in T has a neighbor outside S . Furthermore, the sets S, T can be found in polynomial time in the size of G .*

Henceforth, we will use the following modified version of the expansion lemma.

Lemma 3. (*Modified Expansion Lemma*) *Let r be a positive integer, and let m be the size of the maximum matching in a bipartite graph G with vertex bipartition $X \cup Y$. If $|Y| > rm$, and there are no isolated vertices in Y , then there exists a polynomial algorithm (in the size of G) which returns a partition $X = A_1 \cup B_1, Y = A_2 \cup B_2$, such that B_1 has $|B_1|$ r -stars in B_2 , $E(A_1, B_2) = \emptyset$, and $|A_2| \leq r|A_1|$.*

Proof. If $|Y| \leq rm$, then we may return $A_1 = X, A_2 = Y, B_1 = B_2 = \emptyset$, as $m \leq |X|$ and hence $|Y| \leq r|X|$. Otherwise, apply the Expansion Lemma to get nonempty vertex sets $S \subseteq X, T \subseteq Y$ such that S has $|S|$ r -stars in T and no vertex in T has a neighbor in Y outside S . Let $X' = X \setminus S$ and $Y' = Y \setminus T$. If $G[X' \cup Y']$ has isolated vertices in Y' , move all of them from Y' to T . If $|Y'| \leq r|X'|$, we may return $A_1 = X', A_2 = Y', B_1 = S$, and $B_2 = T$.

So now assume $|Y'| > r|X'|$. In this case, apply the algorithm recursively on $G[X' \cup Y']$ to get a partition $X' = A'_1 \cup B'_1, Y' = A'_2 \cup B'_2$, such that B'_1 has $|B'_1|$ stars in B'_2 , $E(A'_1, B'_2) = \emptyset$, and $|A'_2| \leq r|A'_1|$. Then return $A_1 = A'_1, B_1 = B'_1 \cup S, A_2 = A'_2, B_2 = B'_2 \cup T$. Observe that B_1 has $|B'_1| + |S| = |B_1|$ stars in B_2 , $E(A_1, B_2) \subseteq E(A'_1, B'_2) \cup E(X \setminus S, T) = \emptyset$, and

$|A_2| = |A'_2| \leq r|A'_1| = r|A_1|$, as required. As each iteration reduces $|X|$ by at least 1, we will have to apply less than $|X| + |Y|$ iterations, each of which uses at most one application of the Expansion Lemma, and so the algorithm runs in polynomial time. \square

Proof of Theorem 1. By exhaustively applying Reduction Rule 1, we may assume we have a simplified graph. Let G be a simplified graph in \mathcal{G}_d . Now find a maximal r -star packing of the graph G with q stars. We may assume $q < k$ as otherwise we have a trivial YES-instance. Let S be the set of vertices in this packing, and let $D = V(G) \setminus S$.

For any $u \in D$, let $D[u]$ be the set of vertices $v \in D$ for which there is a path from v to u using only vertices in D - that is, $D[u]$ is the set of vertices in the component of $G[D]$ containing u . As our star-packing is maximal, $d_D(v) < r$ for every $v \in D$. As $G \in \mathcal{G}_d$, every $v \in D[u]$ has a path to u in $G[D]$ with at most $d - 1$ vertices (as otherwise the shortest path in $G[D]$ from v to u is an induced path on at least d vertices). It follows that $|D[u]| \leq 1 + r + r^2 + \dots + r^{d-1} \leq r^d$.

We will now find a partition of S into $Big(S) \cup Small(S)$, and D into $B(D) \cup U(D)$, such that $|B(D)| \leq r^{d+1}|Small(S)|$, and either $Big(S) = U(D) = \emptyset$ or $(Big(S), U(D))$ is a constellation. As $|Small(S)| \leq |S| \leq (r+1)k$, it follows that either $|V(G)| \leq (r+1)k + (r+1)r^{d+1}k$, or we can apply Reduction Rule 2 on $(Big(S), U(D))$.

We will construct $Big(S), Small(S), B(D), U(D)$ algorithmically as described below. Throughout, we will preserve the properties that

1. $|B(D)| \leq |Small(S)|r^{d+1}$,
2. $U(D)$ has no neighbors in $Small(S) \cup B(D)$.

Initially, set $Big(S) = S, U(D) = D, Small(S) = B(D) = \emptyset$.

While $|U(D) \cap N(Big(S))| > r|Big(S)|$, do the following.

If there is a vertex $u \in Big(S)$ such that $|N(u) \cap U(D)| < r$, let $X = \bigcup \{D[v] : v \in N(u) \cap U(D)\}$. Observe that as $|D[v]| \leq r^d$ for all $v \in D$, $|X| < r^{d+1}$. Now set $Small(S) = Small(S) \cup \{u\}$, $Big(S) = Big(S) \setminus \{u\}$, $B(D) = B(D) \cup X$, $U(D) = U(D) \setminus X$. It follows that Property 1 is preserved. Note that no vertex in the new $U(D)$ has a neighbor in X (as all neighbors of X in D lie in X). Similarly no vertex in the new $U(D)$ is adjacent to u (as such a vertex would be in the old $U(D)$ and so would have been added to X). Therefore there are still no edges between the new $U(D)$ and the new $Small(S) \cup B(D)$, and so Property 2 is preserved.

Otherwise (if every vertex $u \in Big(S)$ has $|N(u) \cap U(D)| \geq r$), let H denote the maximal bipartite subgraph of G with vertex partition $Big(S) \cup (U(D) \cap N(Big(S)))$, and apply the Modified Expansion Lemma to H . We will

get a partition $Big(S) = A_1 \cup B_1$ and $U(D) \cap N(Big(S)) = A_2 \cup B_2$ such that $E(A_1, B_2) = \emptyset$, $|A_2| \leq r|A_1|$ and B_1 has $|B_1|$ r -stars in B_2 .

If the Modified Expansion Lemma returns $B_1 = Big(S)$, then we claim that $(Big(S), U(D))$ is a constellation. To see this, firstly note that $|Big(S)|$ has $|Big(S)|$ r -stars in $U(D)$. Secondly, note that since we chose the vertices of a maximal star packing for S , there is no r -star contained in $G[U(D)]$. As $U(D)$ has no neighbors in $Small(S) \cup B(D)$, it follows that there is no r -star intersecting $U(D)$ in $G \setminus Big(S)$. Thus $(Big(S), U(D))$ is a constellation, and the claim is proved. In this case the algorithm stops.

So now assume that the Modified Expansion Lemma returns $Big(S) = A_1 \cup B_1$ with $A_1 \neq \emptyset$. Let $X = \bigcup \{D[v] : v \in N(A_1) \cap U(D)\}$. Note that as $E(A_1, B_2) = \emptyset$ and $|A_2| \leq r|A_1|$, we have $|X| \leq |\bigcup \{D[v] : v \in A_2\}| \leq |A_2|r^d \leq |A_1|r^{d+1}$. Then let $Small(S) = Small(S) \cup A_1$, $Big(S) = Big(S) \setminus A_1$, $B(D) = B(D) \cup X$, $U(D) = U(D) \setminus X$. Note that after this move, we still have that $|B(D)| \leq |Small(S)|r^{d+1}$, and $U(D)$ has no neighbors in $Small(S) \cup B(D)$.

Note that in either case, $|Big(S)|$ strictly decreases, so the algorithm must eventually terminate, either because $(Big(S), U(D))$ is a constellation, or because $|U(D) \cap N(Big(S))| \leq r|Big(S)|$. If $(Big(S), U(D))$ is a constellation, apply Reduction Rule 2 using $(Big(S), U(D))$. This gives us a partition in which $Big(S) = U(D) = \emptyset$. Thus in either case, we have that $|U(D) \cap N(Big(S))| \leq r|Big(S)|$. Note that every vertex $u \in U(D)$ is in $D[v]$ for some $v \in N(S)$ (as otherwise, either $\max\{d(v) : v \in N[u]\} < r$ or $G[D]$ contains an r -star, a contradiction in either case). Moreover such a v must be in $U(D) \cap N(Big(S))$, as there are no edges between $U(D)$ and $Small(S) \cup B(D)$. Thus $|U(D)| \leq r^d |U(D) \cap N(Big(S))| \leq r^{d+1} |Big(S)|$. Then we have $|V(G)| = |S| + |U(D)| + |B(D)| \leq |S| + r^{d+1} |Big(S)| + r^{d+1} |Small(S)| \leq (r^{d+1} + 1)|S| \leq (k - 1)(r + 1)(r^{d+1} + 1) = O(k)$.

4 Proof of Theorem 2

A *split graph* is a graph where the vertex set can be partitioned into a clique and an independent set.

An instance of the well-known \mathcal{NP} -hard problem 3-DIMENSIONAL MATCHING contains a vertex set that can be partitioned into three equally large sets V_1, V_2, V_3 (also called partite sets). Let k denote the size of each of V_1, V_2, V_3 . It furthermore contains a number of 3-sets containing exactly one vertex from each V_i , $i = 1, 2, 3$. The problem is to decide if there exists a set of k vertex disjoint 3-sets (which would then cover all vertices). Such a set of k vertex disjoint 3-sets is called a perfect matching. The 3-sets are also called *edges* (or *hyperedges*).

Theorem 4. *Let $r \geq 3$. It is \mathcal{NP} -hard to decide if the vertex set of a split graph can be partitioned into vertex disjoint copies of $K_{1,r}$.*

Proof. We will reduce from 3-DIMENSIONAL MATCHING. Let \mathcal{I} be an instance of 3-dimensional matching. Let V_1, V_2, V_3 denote the three partite sets of \mathcal{I} and let E denote the set of edges in \mathcal{I} . Let $m = |E|$ and $k = |V_1| = |V_2| = |V_3|$. We will build a split graph $G_{\mathcal{I}}$ as follows. Let $V = V_1 \cup V_2 \cup V_3$ be the vertices of \mathcal{I} . Let X_1 be a set of m vertices and X_2 be a set of $m - k$ vertices and let $X = X_1 \cup X_2$. Let Y be a set of $(m - k)(r - 1)$ vertices and let W be a set of $k(r - 3)$ vertices (if $r = 3$ then W is empty). Let the vertex set of $G_{\mathcal{I}}$ be $V \cup X \cup Y \cup W$.

Add edges such that X becomes a clique in $G_{\mathcal{I}}$. Let each vertex in X_1 correspond to a distinct edge in E and connect that vertex with the 3 vertices in V which belongs to the corresponding edge in E . Furthermore add all edges from X_1 to W . Finally, for each vertex in X_2 add $r - 1$ edges to Y in such a way that each vertex in Y ends up with degree one in $G_{\mathcal{I}}$. This completes the construction of $G_{\mathcal{I}}$.

Clearly $G_{\mathcal{I}}$ is a split graph as X is a clique and $V \cup Y \cup W$ is an independent set. We will now show that the vertex set of $G_{\mathcal{I}}$ can be partitioned into vertex disjoint copies of $K_{1,r}$ if and only if \mathcal{I} has a perfect matching.

First assume that \mathcal{I} has a perfect matching. Let $E' \subseteq E$ denote the edges of the perfect matching. For the vertices in X_1 that correspond to the edges in E' we include the three edges from each such vertex to V as well as $r - 3$ edges to W . This can be done such that we obtain k vertex disjoint copies of $K_{1,r}$ covering all of V and W as well as k vertices from X_1 . Now for each vertex in X_2 include the $r - 1$ edges to Y as well as one edge to an unused vertex in X_1 . This can be done such that we obtain an additional $m - k$ vertex disjoint copies of $K_{1,r}$. We have now constructed m vertex disjoint copies of $K_{1,r}$ which covers all the vertices in $G_{\mathcal{I}}$, as required.

Now assume that the vertex set of $G_{\mathcal{I}}$ can be partitioned into vertex disjoint copies of $K_{1,r}$. As $|V \cup W \cup Y \cup X| = m(r + 1)$ we note that we have m vertex disjoint copies of $K_{1,r}$, which we will denote by \mathcal{K} . As all vertices in Y need to be included in such copies we note that every vertex of X_2 is the center vertex of a $K_{1,r}$. Let \mathcal{K}' denote these $m - k$ copies of $K_{1,r}$. Each $K_{1,r}$ in \mathcal{K}' must include 1 edge from X_2 to X_1 . These $m - k$ edges form a matching, implying that $m - k$ vertices of X_1 also belong to the copies of $K_{1,r}$ in \mathcal{K}' . This leaves k vertices in X_1 that are uncovered and rk vertices in $V \cup W$ that are uncovered. Furthermore, as $V \cup W$ is an independent set, each copy of $K_{1,r}$ in $\mathcal{K} \setminus \mathcal{K}'$ must contain a vertex of X_1 . As $|\mathcal{K} \setminus \mathcal{K}'| = k$ we note that the k copies of $K_{1,r}$ in $\mathcal{K} \setminus \mathcal{K}'$ must include exactly one vertex from X_1 . Also as each vertex in X_1 has exactly three neighbours in V , each such $K_{1,r}$ also contains 3 vertices from V (as V needs to be covered) and therefore $r - 3$ vertices from W . Therefore the

k vertices in X_1 that belong to copies of $K_{1,r}$ in $\mathcal{K} \setminus \mathcal{K}'$ correspond to k edges in E which form a perfect matching in $G_{\mathcal{I}}$.

This completes the proof as we have shown that $G_{\mathcal{I}}$ can be partitioned into vertex disjoint copies of $K_{1,r}$ if and only if \mathcal{I} has a perfect matching. \square

The following lemma is known. We give the simple proof for completeness.

Lemma 4. *No split graph contains an induced path on 5 vertices.*

Proof. Assume G is a split graph where $V(G)$ is partitioned into an independent set I and a clique C . For the sake of contradiction assume that $P = p_0p_1p_2p_3p_4$ is an induced P_5 in G . As I is independent we note that $\{p_0, p_1\} \cap C \neq \emptyset$ and $\{p_3, p_4\} \cap C \neq \emptyset$. As C is a clique there is therefore an edge from a vertex in $\{p_0, p_1\}$ to a vertex in $\{p_3, p_4\}$. This edge implies that P is not an induced P_5 in G , a contradiction. \square

Proof of Theorem 2. By Lemma 4, \mathcal{G}_5 contains all split graphs. The result now follows immediately from Theorem 4. \square

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References

- [1] R. Bar-Yehuda, M. Halldórsson, J. Naor, H. Shachnai, and I. Shapira, Scheduling split intervals, in 30th Annu. ACM-SIAM Symp. on Discrete Algorithms, 2002, pp. 732–741.
- [2] R. Bejar, B. Krishnamachari, C. Gomes, and B. Selman, Distributed constraint satisfaction in a wireless sensor tracking system, Workshop on Distributed Constraint Reasoning, Internat. Joint Conf. on Artificial Intelligence, 2001.
- [3] A. Brandstädt, V.B. Le, and J.P. Spinrad. Graph Classes: A Survey, SIAM, 1999.
- [4] M. Cygan, F.V. Fomin, L. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh, Parameterized Algorithms, Springer, 2015.
- [5] R.G. Downey and M.R. Fellows, Foundations of Parameterized Complexity, Springer, 2013.
- [6] M. Fellows, J. Guo, H. Moser, and R. Niedermeier, J. Comput. Syst. Sci. 77:1141–1158, 2011.

- [7] M. Fellows, P. Heggernes, F. Rosamond, C. Sloper, and J.A. Telle, Finding k Disjoint Triangles in an Arbitrary Graph. In WG'05, Lect. Notes Comput. Sci. 3353:235–244, 2005
- [8] F.V. Fomin, D. Lokshtanov, N. Misra, G. Philip, and S. Saurabh, Hitting forbidden minors: Approximation and Kernelization. In STACS 2011, LIPIcs 9:189–200, 2011.
- [9] J. Guo and R. Niedermeier. Linear problem kernels for NP-hard problems on planar graphs. In ICALP 2007, Lect. Notes Comput. Sci. 4596:375–386, 2007.
- [10] D.G. Kirkpatrick and P. Hell, On the completeness of a generalized matching problem. In 10th STOC, ACM Symposium on Theory of Computing, 240–245, 1978.
- [11] S. Kratsch, Recent developments in kernelization: A survey. Bulletin EATCS, no. 113, 2014.
- [12] D. Lokshtanov, N. Misra, and S. Saurabh, Kernelization - preprocessing with a guarantee. Lect. Notes Comput. Sci. 7370:129–161, 2012.
- [13] E. Prieto and C. Sloper, Looking at the stars, Theor. Comput. Sci. 351:437–445, 2006.
- [14] J. Wang, D. Ning, Q. Feng, and J. Chen, An improved parameterized algorithm for a generalized matching problem, In TAMC'08, Lect. Notes Comput. Sci., 4978:212–222, 2008.

A Proof of Theorem 3

Note that \mathcal{G}_4 is the family of *cographs* [3]. It is well-known [3] that any non-trivial (i.e., with at least two vertices) cograph G is either disconnected or its complement is disconnected. Below let n denote the order of G and let m denote the size of G . The following lemma is well-known.

Lemma 5. *For any graph G , we can in time $O(n^2)$ find the connected components of G and the connected components of the complement of G .*

Lemma 6. *For any $G \in \mathcal{G}_4$ and any $s \geq 1$ we can in time $O(n^2)$ find a set of s vertices, say S , in G such that $|N[S]|$ is maximum possible.*

Proof. Let C_1, C_2, \dots, C_l be the connected components of G ($l \geq 1$). Assume first that all the components are non-trivial. As any induced subgraph of a cograph is also a cograph we note that the complement of each C_i is disconnected. Therefore for each $i = 1, 2, \dots, l$ there exists a non-trivial (each part is non-empty) partition (X_i, Y_i) of $V(C_i)$ such that all edges exist between X_i and Y_i in G . Let m_i be maximum degree of a vertex in C_i for each $i = 1, 2, \dots, l$.

The maximum number of vertices we can add to $N[S]$ by adding one vertex from C_i is $m_i + 1$ and the maximum number of vertices added to $N[S]$ by adding two vertices from C_i is $|V(C_i)|$ as we can add a vertex from X_i and one from Y_i . Therefore the maximum possible $|N[S]|$ is the sum of the s largest numbers in the set $m_1 + 1, m_2 + 1, \dots, m_s + 1, (|V(C_1)| - m_1 - 1), (|V(C_2)| - m_2 - 1), \dots, (|V(C_l)| - m_l - 1)$. Furthermore it is easy to find the actual set S .

It is not hard to modify the proof above for the case when some C_i 's are trivial. \square

Now we are ready to prove the main result of this appendix.

Proof of Theorem 3: Let $G \in \mathcal{G}_4$ and let $r \geq 3$ be arbitrary. First assume that G is connected, which implies that the complement of G is disconnected. Let X and Y partition $V(G)$ such that all edges exist between X and Y in G . We now consider two cases.

Case 1: $|X| > r|Y|$ or $|Y| > r|X|$. Without loss of generality, assume that $|X| > r|Y|$. In this case we recursively find the maximum number of r -stars we can pack into $G[X]$. Let m_x be the maximum number of r -stars in $G[X]$. If $(r + 1)m_x + (r + 1)|Y| \leq n$, then the optimal answer is that we can pack $m_x + |Y|$ r -stars into G as we can always find $|Y|$ r -stars with centers in Y and not touching the m_x r -stars we already found in $G[X]$. If

$(r+1)m_x + (r+1)|Y| > n$, then the optimal solution is $\lfloor n/(r+1) \rfloor$ r -stars as we can pick $|Y|$ r -stars touching as few of the m_x r -stars in $G[X]$ as possible and then pick as many of the m_x r -stars that are left untouched. This completes this case.

Case 2: $|X| \leq r|Y|$ and $|Y| \leq r|X|$. Let $x = |X|$ and $y = |Y|$ and define a and b as follows:

$$a = \frac{ry - x}{r^2 - 1} \text{ and } b = \frac{rx - y}{r^2 - 1}$$

Let $a' = \lfloor a \rfloor = a - \epsilon_a$ and $b' = \lfloor b \rfloor = b - \epsilon_b$. We will first show that we can find $a' + b'$ r -stars such that a' of the r -stars have the center in X and all leaves in Y and b' of the r -stars have the center in Y and all leaves in X . This is possible due to the following:

$$\begin{aligned} a'r + b' &= (a - \epsilon_a)r + (b - \epsilon_b) \\ &= r \frac{ry - x}{r^2 - 1} + \frac{rx - y}{r^2 - 1} - (r\epsilon_a + \epsilon_b) \\ &= y - (r\epsilon_a + \epsilon_b) \end{aligned}$$

And, analogously,

$$b'r + a' = x - (r\epsilon_b + \epsilon_a)$$

As $0 \leq \epsilon_a < 1$ and $0 \leq \epsilon_b < 1$ we note that we cover all vertices in G except $r\epsilon_a + \epsilon_b + r\epsilon_b + \epsilon_a = (r+1)(\epsilon_a + \epsilon_b)$. Therefore the number of vertices we cannot cover by the r -stars above is strictly less than $2(r+1)$. If $(r+1)(\epsilon_a + \epsilon_b) < r+1$ then we have an optimal solution (covering all vertices except at most r), so assume that $(r+1)(\epsilon_a + \epsilon_b) \geq r+1$.

Clearly the optimal solution is either $a' + b'$ or $a' + b' + 1$. As we already have a solution with $a' + b'$ r -stars we will now determine if there is a solution with $a' + b' + 1$ r -stars.

If some vertex, say w_x , in X has degree at least r in $G[X]$, then there is indeed a solution with $a' + b' + 1$ r -stars, because of the following. As $(r+1)(\epsilon_a + \epsilon_b) \geq r+1$ we must have $\epsilon_a > 0$ and $\epsilon_b > 0$, which implies that we can pick an r -star with center in $w_x \in X$ and with at most $r\epsilon_b + \epsilon_a - 1$ leaves in X and at most $r\epsilon_a + \epsilon_b$ leaves in Y . Once this r -star has been picked it is not difficult to pick an additional a' r -stars with centers in X (and leaves in Y) and b' r -stars with centers in Y (and leaves in X), due to the above. Therefore we may assume no vertex in X has degree at least r in $G[X]$. Analogously we may assume that no vertex in Y has degree at least r in $G[Y]$.

If there exists $a' + 1$ vertices S_X in X such that $|N[S_X] \cap X| \geq a' + 1 + r - (r\epsilon_a + \epsilon_b)$, then proceed as follows. We can create $a' + 1$ stars in $G[X]$ such that they together have exactly $r - (r\epsilon_a + \epsilon_b)$ non-centers. By the above each

star has less than r leaves, so we can expand these $a' + 1$ stars to r -stars by adding leaves from Y . This uses up $a' + 1 + r - (r\epsilon_a + \epsilon_b)$ vertices from X and $(a' + 1)r - (r - (r\epsilon_a + \epsilon_b))$ vertices from Y . Adding an additional b' stars with the center in Y and all leaves in X uses up b' vertices from Y and rb' vertices from X . Therefore we have used $b' + a'r + (r\epsilon_a + \epsilon_b) = y$ vertices from Y and the following number of vertices from X ,

$$a' + rb' + 1 + r - (r\epsilon_a + \epsilon_b) = x - (r\epsilon_b + \epsilon_a) + 1 + r - (r\epsilon_a + \epsilon_b) = x + 1 + r - (r + 1)(\epsilon_a + \epsilon_b)$$

As $(r + 1)(\epsilon_a + \epsilon_b) \geq r + 1$ we note that we use at most x vertices from X and we have a solution with $a' + b' + 1$ r -stars. Analogously if there exists $b' + 1$ vertices S_Y in Y such that $|N[S_Y] \cap X| \geq b' + 1 + r - (r\epsilon_b + \epsilon_a)$, we obtain $a' + b' + 1$ r -stars. By applying Lemma 6 to $G[X]$ and $G[Y]$ we can decide the above in polynomial time.

We may therefore assume that no such S_X or S_Y exist. We will now show that $a' + b'$ is the optimal solution. For the sake of contradiction assume that we have a^* r -stars with centers in X and b^* r -stars with centers in Y , such that they are vertex disjoint and $a^* + b^* = a' + b' + 1$. Without loss of generality we may assume that $a^* \geq a' + 1$. The a^* r -stars with centers in X all have at least one leaf in Y as the maximum degree in $G[X]$ is less than r . Furthermore by the above (S_X does not exist) any $a' + 1$ r -stars with centers in X have more than $r(a' + 1) - (r - (r\epsilon_a + \epsilon_b))$ leaves in Y . Therefore we use strictly more than the following number of vertices in Y .

$$r(a' + 1) - (r - (r\epsilon_a + \epsilon_b)) + (a^* + b^* - (a' + 1)) = ra' + r\epsilon_a + \epsilon_b + b' = y$$

This contradiction implies that the optimal solution is $a' + b'$ in this case. This completes the case when G is connected.

Finally assume that G is disconnected. In this case we recursively solve the problem for each connected component, which can be added together to get an optimal solution for G . It is not difficult to see that the above can be done in polynomial time. \square